TEST 1

Shef Scholars

March 2024

1 Problems

Make sure to block out enough time for the test.

You may use your phone or mechanical clock as a timer.

Once you begin the exam, do not use the internet until you've finished the exam. Do not consult textbooks, chatbots, or outside sources. Any cheating may result in your application automatically being rejected and you being unable to apply to future iterations of Shef's Scholars or even other events organized by Shefs of Problem Solving.

Write up your solutions in English and put your email address (the same one you're putting in your application) on top of the paper on each submitted page.

The 3 of the problems are of roughly equal difficulty with 1 problem being a bit harder than other problems in all 3 exams. Note that problem difficulty is ultimately subjective and what might be the easiest problem in my mind might be the hardest for you and vice versa. The first problem in all 3 exams is considered to be roughly the easiest.

You have:

1. 180 minutes (3 hours) of exam time for the Begging Level exam which in on page X
2. 210 minutes (3 and a half hours) of exam time for the Apprentice Level exam which is on page 4
3. 240 minutes (4 hours) of exam time for the Machine level exam which is on page 6
4. 270 minutes (4 and a half hours) of exam time for the Shef level exam which is on page 8

Unless you are submitting your exam in word or LaTeXthe write up is part of the time you have for the problem. If you are submitting your exam in word or LaTeXplease write up the solutions on paper first and then rewrite them on LaTeXor word. Feel free not to submit papers you don't think are important. Note how long you took the exam on one of the papers.

Feel free to use the bathroom any amount of time you like during the exam and eat and drink as much as you like. My advice would be to treat this like a competition exam, so that you can practice your problem solving skills under some artificial exam pressure.

Label the problems and problem pages on top of every paper you submit.

Note that the exam isn't the determining factor in your application. We will try to look at your application as a whole, which may include an interview.

Just focus on doing your best !

Finally, enjoy the problems, they are fun :)

The Beginner

The Begginer Problems

1. Consider the positive integer $n = 7+7^2+7^3+...+7^{2023}$
   (a) Show that the remainders of the divisions of 7^{2024} by 6 and by 48 are equal.
   (b) Determine the last two decimal digits of the number $6n$
2. Prove that numbers $1, 2, \ldots, 16$ can be placed on a line such that sum of any two neighboring numbers is perfect square.
3. Let a, b, and c be real numbers that satisfy $a+b+c = 0$ and $a^2 + b^2 + c^2 = 1$. Determine the value of $a^4 + b^4 + c^4$.
4. Let ABC be an acute angled triangle with $AC > AB$. Let A', B', C' be the midpoints of sides BC, CA, AB, respectively. Let D be the foot of the altitude from A to BC. Prove that $\angle DC'A' = \angle DB'A' = \angle ABC - \angle ACB$.

The Apprentice

The Apprentice Problems

1. If the positive integers a, b, c satisfy the inequalities $a > b > c$ and $12b > 13c > 11a$, show that $a+b+c \geq 56$.

2. Two children, Shef and Machine, play a game several times, in which the winner receives x points, and the loser y points (x and y are nonnegative integers, with $x > y$, and in any game of the children is the winner and other is the loser). The final score 147 to 123, in Shef's favour. The Machine has won 6 games. Determine the numbers x and y.

3. Let ABC be a triangle with $AB = AC$ and $\angle BAC = 100$. Let BD be the angle bisector of the angle $\angle ABC$ with $D \in AC$, and E a point on line BD such that D lies between B and E such that $BE = BC$. Let F be on segment BC such that $AB = BF$. Prove that the lines AC and EF are ortogonal.

4. Let $n \geq 2$ be a positive integer. We say that a positive integer $\overline{a_1 a_2 ... a_n}$ is *good* if

$$\overline{a_1 a_2 ... a_n} = a_1 \cdot a_2 \cdot ... \cdot a_n + a_1 + a_2 + ... + a_n$$

Find all *good* positive integers.

The Machine

The Machine Problems

1. Consider the set

$$M = \left\{\frac{a}{\overline{ba}} + \frac{b}{\overline{ab}} \mid a, b \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}\right\}$$

   (a) Show that the set M contains no integer

   (b) Find the smallest and largest element of M.

2. Find all non-negative integers x, y and z such that $x^3 + 2y^3 + 4z^3 = 9!$

3. Find positive integers n and $a_1, a_2, \ldots, a_n$ such that

$$a_1 + a_2 + \ldots a_n = 1979$$

   and the product $a_1 a_2 \ldots a_n$ as large as possible.

4. In acute-angled triangle ABC, BH is the altitude of the vertex B. The points D and E are midpoints of AB and AC respectively. Suppose that F be the reflection of H with respect to ED. Prove that the line BF passes through circumcenter of ABC.

The Shef

The Shef Problems

1. Show that for every integer $n \geq 3$ there exists positive integers $[x_1, 2x_2, x_3, \cdots, x_n$ pairwise different so that $\{2, n\} \subset \{x_1, x_2, \ldots, x_n\}$ and

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1$$

2. Let n and k be two positive integers such that $1 \leq n \leq k$. Prove that if $d^k + k$ is a prime number for each positive divisor d of n, then $n + k$ is a prime number.

3. In triangle ABC, I is the incenter. We have chosen points P, Q, R on segments IA, IB, IC respectively such that $IP \cdot IA = IQ \cdot IB = IR \cdot IC$. Prove that the points I and O belong to Euler line of triangle PQR where O is circumcenter of ABC.

4. Let A and B be two finite sets. Find the number of functions $f : A \to B$ satisfying the property that there exist two functions $g : A \to B$ and $h : B \to A$ such that $g(h(x)) = x, \forall x \in B$ and $h(g(x)) = f(x), \forall x \in A$.